Appendix B

$$2\sum_{q=1}^{\infty}\frac{J_n(u_{nq}a)J_n(u_{nq}y)}{u_{nq}^2\left(u_{nq}^2+\mu^2\right)J_{n+1}^2(u_{nq})}=\frac{1}{\mu^2}\left|\begin{array}{ll} y^n/2n\{a^{-n}-a^n\}+I_n(\mu y)\{I_n(\mu a)K_n(\mu)/I_n(\mu)-K_n(\mu a)\}, & 0\leqslant y\leqslant a\leqslant 1\\ a^n/2n\{y^{-n}-y^n\}+I_n(\mu a)\{I_n(\mu y)K_n(\mu)/I_n(\mu)-K_n(\mu y)\}, & 0\leqslant a\leqslant y\leqslant 1\end{array}\right|$$

$$2\sum_{q=1}^{\infty}\frac{J_n^2(u_{nq}a)}{\left(u_{nq}^2+\mu^2\right)^2J_{n+1}^2(u_{nq})}=-\frac{I_n^2(\mu a)}{2\mu^2I_n^2(\mu)}+\frac{a}{\mu}\left\{I_n'(\mu a)I_n(\mu a)\frac{K_n(\mu)}{I_n(\mu)}-\frac{1}{2}\{K_n(\mu a)I_n'(\mu a)+I_n(\mu a)K_n'(\mu a)\}\right\}$$

$$2\sum_{q=1}^{\infty}\frac{v_{nq}^2}{v_{nq}^2-n^2}\frac{J_n(v_{nq}a)J_n(v_{nq}x)}{v_{nq}^2\left(v_{nq}^2+\mu^2\right)J_n^2(v_{nq})}=\frac{1}{\mu^2}\left|\begin{array}{ll}\dfrac{x^n}{2n}\{a^n+a^{-n}\}+I_n(\mu x)\left\{I_n(\mu a)\dfrac{K_n'(\mu)}{I_n'(\mu)}-K_n(\mu a)\right\}, & 0\leqslant x\leqslant a\leqslant 1\\ \dfrac{a^n}{2n}\{x^n+x^{-n}\}+\left\{I_n(\mu x)\dfrac{K_n'(\mu)}{I_n'(\mu)}-K_n(\mu x)\right\}I_n(\mu a), & 0\leqslant a\leqslant x\leqslant 1\end{array}\right|$$

$$2\sum_{q=1}^{\infty}\frac{v_{nq}^4}{v_{nq}^2-n^2}\frac{J_n'^2(v_{nq}a)}{\left(v_{nq}^2+\mu^2\right)^2J_n^2(v_{nq})}=\left(1+\frac{n^2}{\mu^2}\right)\frac{I_n'^2(\mu a)}{2I_n'^2(\mu)}+\mu a\left(1+\frac{n^2}{\mu^2a^2}\right)$$
$$\cdot\left\{I_n(\mu a)I_n'(\mu a)\frac{K_n'(\mu)}{I_n'(\mu)}-\frac{1}{2}\{I_n(\mu a)K_n'(\mu a)+I_n'(\mu a)K_n(\mu a)\}\right\}.$$

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A General Reciprocity Theorem

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***Abstract*—A general reciprocity theorem based on the Onsager relations is developed which applies to all causal and linear media, including those whose ac susceptibilities depend on an applied dc magnetic field and on the dc drift velocity of charge carriers. Applications are made to the scattering matrix for microwave junctions and to the mode orthogonality relations for uniform and periodic waveguides.**

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I. Introduction

ONE OF THE BASIC theorems of electromagnetic theory is the reciprocity theorem. There has been a long history of contributions to its development [1]–[6]. It is the purpose of this contribution to extend the range of applicability of the reciprocity theorem and to provide a physical basis for it through the Onsager relations. The requirements are only that the media in the region under consideration be causal and linear; they may be either passive or active. In particular, media whose ac suscept-

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ibilities may depend on an applied dc magnetic field and (or) the dc drift velocity of charge carriers are included. Applications of this general reciprocity theorem to the form of the scattering matrix for microwave junctions, and to the orthogonality of modes in uniform and periodic waveguides, are noted.

II. General Reciprocity Theorem

The physical basis of the general reciprocity theorem is first established. Consider the excitation at frequency ω of a time invariant medium by a force $\boldsymbol{F}$ with a resulting response $\boldsymbol{R}$. If the medium is causal and linear, $\boldsymbol{R}=\chi(\omega)\cdot\boldsymbol{F}$, where $\chi(\omega)$ is the susceptibility. For example, $\boldsymbol{F}$ might be the electric field, $\boldsymbol{R}$ the electric polarization, and $\chi(\omega)$ the electric susceptibility of a dielectric. More generally, χ may be dependent on an applied dc magnetic field $\boldsymbol{H}_0$ and (or) the dc drift velocity $\boldsymbol{v}_0$ of charge carriers in the medium $\chi(\omega,\boldsymbol{H}_0,\boldsymbol{v}_0)$. For example, a ferrite with a dc applied magnetic field $\boldsymbol{H}_0$ has an ac magnetic susceptibility which depends on $\boldsymbol{H}_0$, while an electron beam has susceptibilities which, in general, will depend on both the dc drift velocity of the electrons and the dc magnetic focusing field.

Onsager [7] applied the principle of the time-reversal invariance of the microscopic equations of motion to show that any macroscopic susceptibility of a causal linear medium must satisfy the relation

$$\chi(\omega)=\tilde{\chi}(\omega).$$

Here, $\tilde{\chi}$ denotes the transpose of χ. Including the possible dependence on the dc magnetic field [8], [9] and on the dc drift velocity of charge carriers [10], one finds that

$$\chi(\omega,-\boldsymbol{H}_0,-\boldsymbol{v}_0)=\tilde{\chi}(\omega,\boldsymbol{H}_0,\boldsymbol{v}_0). \tag{1}$$

More precisely, (1) applies when $\boldsymbol{F}$ and $\boldsymbol{R}$ both change sign, or neither changes sign, under time reversal. If one changes sign under time reversal and the other does not, then a minus sign should be inserted on the right side of (1). Note that both $\boldsymbol{H}_0$ and $\boldsymbol{v}_0$ may vary in magnitude and direction through the medium. Also note that (1) is independent of the presence of any dc electric field, except in so far as it affects the dc drift velocity of the charge carriers. Equation (1) applies to the susceptibilities for all media, both passive and active, so long as they are causal and linear.

A common representation of the reciprocity theorem is given in

$$\iint[\boldsymbol{E}\times\boldsymbol{H}'-\boldsymbol{E}'\times\boldsymbol{H}]\cdot d\boldsymbol{S}=\iiint[\boldsymbol{E}'\cdot\boldsymbol{J}_s-\boldsymbol{E}\cdot\boldsymbol{J}_s']\,dV. \tag{2}$$

Here, $\boldsymbol{E}$, $\boldsymbol{H}$ and $\boldsymbol{E}'$, $\boldsymbol{H}'$ are two independent electromagnetic field configurations produced by source current densities $\boldsymbol{J}_s$ and $\boldsymbol{J}_s'$, respectively, at frequency ω. The surface integral on the left side of (2) is over any surface containing the source current densities included in the volume integral on the right side.

With (1) established, the range of applicability of the reciprocity theorem represented by (2) is very broad, and it applies to any region containing causal and linear media. $\boldsymbol{E}$, $\boldsymbol{H}$, and $\boldsymbol{J}_s$ are the electromagnetic field and source current density (frequency ω) for a particular excitation of the region. $\boldsymbol{E}'$, $\boldsymbol{H}'$, and $\boldsymbol{J}_s'$ are the values for an independent excitation of the region (frequency ω), but with the applied dc magnetic field and dc drift velocity of the charge carriers reversed. For example, the reciprocity theorem applies to semiconductor, ferrite, magnetoelectric, electron beam, plasma, maser, and laser media in the linear regime.

Using this result for the reciprocity theorem, and following the procedure of [3], one can show that the scattering matrix for a microwave junction must satisfy

$$\boldsymbol{S}(\omega,-\boldsymbol{H}_0,-\boldsymbol{v}_0)=\tilde{\boldsymbol{S}}(\omega,\boldsymbol{H}_0,\boldsymbol{v}_0). \tag{3}$$

Equation (3) applies to any microwave junction containing causal and linear media, and junctions using any, or several, of the media listed above are included.

III. Mode Orthogonality in Waveguides

Based on the general reciprocity theorem, a general mode orthogonality relation for uniform waveguides can be derived

$$\iint[\boldsymbol{E}_{Tm}\times\boldsymbol{H}_{Tn}'-\boldsymbol{E}_{Tn}'\times\boldsymbol{H}_{Tm}]\cdot\boldsymbol{a}_z\,dx\,dy=0. \tag{4}$$

The unprimed symbols refer to the modal electromagnetic field of the mth mode of the original waveguide, and the primed symbols refer to the modal electromagnetic field of the nth mode of the complementary waveguide (dc magnetic field and dc drift velocity of the charge carriers reversed).

A general-mode orthogonality relation for periodic waveguides can also be derived. This is discussed in some detail here, because the author has been unable to locate in the published literature any prior derivation of the mode orthogonality relation for periodic waveguides.

From Floquet's theorem, the fields of the mth mode of a periodic waveguide with period L can be written as

$$\begin{aligned}\hat{\boldsymbol{E}}_m(x,y,z)&=\boldsymbol{E}_m(x,y,z)\exp[-\gamma_m z]\\&=\sum_p \boldsymbol{E}_{mp}(x,y)\exp[-\gamma_{mp}z]\\\hat{\boldsymbol{H}}_m(x,y,z)&=\boldsymbol{H}_m(x,y,z)\exp[-\gamma_m z]\\&=\sum_p \boldsymbol{H}_{mp}(x,y)\exp[-\gamma_{mp}z].\end{aligned}$$

Here, $\boldsymbol{E}_{mp},\boldsymbol{H}_{mp}$ are the space harmonic components, and $\gamma_{mp}=\gamma_m+j2\pi p/L$ is the propagation constant of the pth space harmonic of the mth mode. These fields satisfy the Floquet condition, $\boldsymbol{E}_m(x,y,z+L)=\boldsymbol{E}_m(x,y,z)$, etc. A similar set of fields with primed symbols is defined for the complementary waveguide with $\boldsymbol{H}_0$ and $\boldsymbol{v}_0$ reversed.

Set $\boldsymbol{P}_{mn}=[\hat{\boldsymbol{E}}\times\hat{\boldsymbol{H}}_n'-\hat{\boldsymbol{E}}_n'\times\hat{\boldsymbol{H}}_m]$ and consider $\nabla\cdot\boldsymbol{P}_{mn}$. Using (1) it is easy to show that $\nabla\cdot\boldsymbol{P}_{mn}=0$. Integrating over one period of the waveguide (from z_1 to z_1+L, where z_1 is arbitrary) and using the divergence theorem, one obtains

$$\iiint\nabla\cdot\boldsymbol{P}_{mn}\,dV=\iint\boldsymbol{P}_{mn}\cdot\boldsymbol{a}_n\,dS=0 \tag{5}$$

where $\boldsymbol{a}_n$ is a unit vector normal to the surface bounding the volume. The contribution to the surface integral from the transverse boundaries of the waveguide is zero because of the boundary conditions for the modal fields. This leaves two integrals over the cross sections at z_1 and z_1+L. Substituting in the modal fields and using the Floquet condition, (5) can be written as

$$\exp\left[-(\gamma_m+\gamma_n')z_1\right]\left(\exp\left[-(\gamma_m+\gamma_n')L\right]-1\right)\cdot\iint\left[\boldsymbol{E}_{Tm}\times\boldsymbol{H}_{Tn}'-\boldsymbol{E}_{Tn}'\times\boldsymbol{H}_{Tm}\right]\cdot\boldsymbol{a}_z\,dxdy=0.$$

Here the integral is over the cross section of the waveguide at z_1, and it involves only the transverse components of the modal fields. For $\gamma_n'\neq-\gamma_m$,

$$\iint\left[\boldsymbol{E}_{Tm}\times\boldsymbol{H}_{Tn}'-\boldsymbol{E}_{Tn}'\times\boldsymbol{H}_{Tm}\right]\cdot\boldsymbol{a}_z\,dxdy=0. \qquad (6)$$

This is the general-mode orthogonality relation for periodic waveguides. More precisely, (6) holds for $\gamma_n'\neq-\gamma_m+j2\pi q/L$, where q is any integer. If $\gamma_n'\neq-\gamma_m$, one would not expect $\gamma_n'=-\gamma_m+j2\pi q/L$ except, possibly, at isolated frequencies.

An alternative form for this mode orthogonality relation involving the space harmonic components can be derived. Integrate (6) in z over one period of the waveguide, expressing $\boldsymbol{E}_{Tm}$ in terms of the space harmonic components, $\boldsymbol{E}_{Tm}=\sum\boldsymbol{E}_{Tmp}\exp\left[-j2\pi pz/L\right]$, etc. From the orthogonality of the exponential functions on any interval of length L, one obtains ($\gamma_n'\neq-\gamma_m$)

$$\sum_p\iint\left[\boldsymbol{E}_{Tmp}\times\boldsymbol{H}_{Tn,-p}'-\boldsymbol{E}_{Tn,-p}'\times\boldsymbol{H}_{Tmp}\right]\cdot\boldsymbol{a}_z\,dxdy=0. \quad (7)$$

This is the second form of the general-mode orthogonality relation for periodic waveguides. These mode orthogonality relations, (6) and (7), hold for all periodic waveguides whose media are causal and linear. The media may be inhomogeneous, anisotropic, and passive or active.

IV. Summary

In summary, this paper has established that the physical basis of a general reciprocity theorem is, through the Onsager relations, the time-reversal invariance of the microscopic equations of motion for linear media. The reciprocity theorem applies to all causal and linear media, including those whose ac susceptibilities depend on an applied dc magnetic field and on the dc drift velocity of charge carriers. The reciprocity theorem applies, for example, to semiconductor, ferrite, magnetoelectric, electron beam, plasma, maser, and laser media in the linear regime; the media may be either passive or active. As a direct consequence, the general scattering matrix relation given in (3) holds for all microwave junctions containing such media.

Application of the reciprocity theorem yields the general mode orthogonality relation for uniform waveguides given in (4). In addition, the general mode orthogonality relations for periodic waveguides given in (6) and (7) were derived. Again, the physical basis for all of these mode orthogonality relations is the time-reversal invariance of the microscopic equations of motion for linear media (through the Onsager relations).

The final comment concerns the general Onsager relations given in (1). Some anisotropic media may exhibit spatial dispersion as well as temporal dispersion, at least in some wavelength ranges. That is, one or more of the susceptibilities may show an explicit dependence on the phase constant as well as the frequency. Examples of such media are plasmas [11], ferrites in the spin wave region [12], and electron beams [13]. For regions containing spatially dispersive media, the susceptibilities must satisfy the relation

$$\chi(\omega,-\gamma,-\boldsymbol{H}_0,-\boldsymbol{v}_0)=\tilde{\chi}(\omega,\gamma,\boldsymbol{H}_0,\boldsymbol{v}_0)$$

in order to obtain the general reciprocity theorem.

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